Assessing Optimal Assignment under Uncertainty: An Interval-based Algorithm

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Problem Addressed

- Multi-robot task allocation.
- Uncertainty in terms of utility-estimation.
- Optimal assignment possible?
- Complexity of the solution.
Solution Abstract

- Modified Hungarian Algorithm: Interval Hungarian Algorithm.
- Time complexity: $O(n^4)$.
- Optimal assignment maintained under uncertainty!!
Hungarian Algorithm

**Algorithm II.1 The Hungarian Algorithm**

**Input:**
A valid $n \times n$ assignment matrix represented as the equivalent complete weighted bipartite graph $G = (X, Y, E)$, where $|X| = |Y| = n$.

**Output:**
A perfect matching, $M$.

1. Generate an initial labelling $l$ and matching $M$ in $G_e$.
2. If $M$ perfect, terminate algorithm. Otherwise, randomly pick an exposed vertex $u \in X$. Set $S = \{u\}$, $T = \emptyset$.
3. If $N(S) = T$, update labels:
   \[ \delta = \min_{x \in S, y \in Y - T} \{ l(x) + l(y) - w(x, y) \} \]
   \[ l'(v) = \begin{cases} 
   l(v) - \delta & \text{if } v \in S \\
   l(v) + \delta & \text{if } v \in T \\
   l(v) & \text{otherwise} 
   \end{cases} \]

4. If $N(S) \neq T$, pick $y \in N(S) - T$.
   (a) If $y$ exposed, $u \rightarrow y$ is augmenting path. Augment $M$ and go to step 2.
   (b) If $y$ matched, say to $z$, extend Hungarian tree: $S = S \cup \{z\}$, $T = T \cup \{y\}$, and go to step 3.

* Definitions:
  - Equality graph $G_e = \{e(x, y) : l(x) + l(y) = w(x, y)\}$
  - Neighbor $N(u)$ of vertex $u \in X$: $N(u) = \{v : e(u, v) \in G_e\}$.
How this works

(a) Table: 

<table>
<thead>
<tr>
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<th>t₁</th>
<th>t₂</th>
<th>t₃</th>
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<tbody>
<tr>
<td>r₁</td>
<td>7</td>
<td>4</td>
<td>3</td>
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<tr>
<td>r₂</td>
<td>6</td>
<td>8</td>
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<tr>
<td>r₃</td>
<td>9</td>
<td>4</td>
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(b) Graph: 

(c) Graph: 

(d) Table: 

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Definitions

- **Matching**: Given a graph $G = (V,E)$, a matching $M$ in $G$ is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex.

- **Matched**: A vertex is matched if it is an endpoint of one of the edges in the matching. Otherwise the vertex is unmatched.
The allowable intervals for matched edge weights is analyzed as follows: for any such edge $e_m(r_\alpha, t_\beta)$, the interval can be described as $[w_{m\alpha\beta} - \varepsilon_m, +\infty)$, where $w_{m\alpha\beta}$ is the edge weight of $e_m(r_\alpha, t_\beta)$ and $\varepsilon_m$ is the tolerance margin that the weight can decrease without violating the optimality of the current matching solution. It is safe to increase the weight as this is a maximization problem. We say a matched edge is hidden if its weight has decreased so as to no longer form part of a matching solution.
Lemma 4.1: With the resultant matching solution $M_0$ and bipartite graph of Hungarian algorithm, if a matched edge $e_m(r_\alpha, t_\beta)$ is hidden, then the Hungarian algorithm can be completed with one iteration rooted at exposed node $r_\alpha$. When a new perfect matching solution $M'$ exists, the labeling reduction of the root $r_\alpha$ satisfies $l(r_\alpha) - l'(r_\alpha) = m_0 - m'$. 
Theorem 4.2 (Matched Edge Interval): Hiding a matched edge from the Hungarian solution leads to a new solution, and the labeling reduction $\varepsilon_m$ at the root of the Hungarian tree is the tolerance margin for this element, i.e., the safe interval for matched edge $e_m(r_\alpha, t_\beta)$ is $[w_{m\alpha\beta} - \varepsilon_m, +\infty)$. 
Bipartite Graph Solution
Lemma 4.3: In the resultant bipartite graph of the Hungarian algorithm, the weight of any unmatched edge $e_u(r_x, t_y)$ can be increased to the sum of two associated labeling values $l(r_x) + l(t_y)$ without affecting the assignment optimum.
Theorem 4.4 (Unmatched Edge Interval): Any unmatched edge $e_u(r_\alpha, t_\beta)$ in the Hungarian resultant bipartite graph, has interval tolerance margin $\varepsilon_u = m_0 - (m_a + l(r_\alpha) + l(t_\beta))$, where $m_0$ is the optimum of the original solution, and $m_a$ is the optimum of the auxiliary bipartite graph associated with $e_u(r_\alpha, t_\beta)$. The allowable interval for edge $e_u(r_\alpha, t_\beta)$ is $(-\infty, m_0 - m_a]$. 
Complexity Analysis

- In Ga, unmatched edge number is \((n-1)\)
- Computation needed → \(O((n-1)^2)\).
- Total number of unmatched edges \((n^2-n)\).
- Total complexity → \(O((n^2-n) \times (n-1)^2) = O(n^4)\).
Uncertainty of a Single Interval

Theorem 5.1 (Uncertainty of a Single Interval): With regard to any specific single utility value, assuming other utilities are certain, the perfect matching solutions are identical if and only if any specific utility is within its allowable interval.
Theorem 5.2 (Uncertainty of Interrelated Intervals):
Given a set of \( n \) interrelated edges, assume \( e_m \) is the matched edge with interval \([w_m - \varepsilon_m, +\infty)\), and \( e_{ui} \) are unmatched edges with intervals \((-\infty, w_{ui} + \varepsilon_{ui}]\), \((i = 1, 2, ..., n - 1)\), then for any \( \varepsilon' \leq \varepsilon_m \), the weight of \( e_m \) can be safely substituted with \( w_m - \varepsilon' \), and the interval for \( e_{ui} \) becomes \((-\infty, w_{ui} + \varepsilon_{ui} - \varepsilon']\), \((i = 1, 2, ..., n - 1)\).
Determining Probability

1) Determine $\varepsilon_{\text{min}}$ from all interrelated edges:
   
   \[
   \varepsilon_{\text{min}} = \min(\varepsilon_m, \varepsilon_{ui}), \quad (i = 1, 2, \ldots, n - 1)
   \]

2) Determine each interrelated interval $I_i$:
   
   \[
   I_i = \begin{cases} 
   \left[w_m - k \cdot \varepsilon_{\text{min}}, +\infty\right) \\
   (-\infty, w_{ui} + \varepsilon_{ui} - k \cdot \varepsilon_{\text{min}}], \quad (i = 1, 2, \ldots, n - 1)
   \end{cases}
   \]

   *$I_0$ represents interval for the matched edge. $k$ is an empirical coefficient and $k \in [0, 1]$ which effects the degree to which the matched and unmatched interval’s are scaled.

3) Determine probability:
   
   \[
   P_{I_i} = \begin{cases} 
   \int_{\xi_m^i}^{+\infty} f(x), \quad (\xi_m^i = w_m - k \cdot \varepsilon_{\text{min}}) \\
   \int_{-\infty}^{\xi_{ui}^i} f(x), \quad (\xi_{ui}^i = w_{ui} + \varepsilon_{ui} - k \cdot \varepsilon_{\text{min}})
   \end{cases}
   \]

4) Determine reliability level:
   The assignment is reliable when $P_{I_i} \geq T$, and unreliable otherwise.
Fig. 6. Multi-robot task assignment using localization. (a) Planned path with wavefront driver; (b) Possible localization results of robot #3.
Result with Robot 1
Unreliable Assignment
Real Robot Experiment

- Significant measurement error.
  - Greater uncertainty.
  - Slower convergence.
Localization with iRobot

Fig. 10. Physical robot localizing in a floor legend
Hypothesis Probability with iRobot

(a) Path cost distribution from the simulated robot after 117 seconds.  
(b) Path cost distribution from the physical robot after 319 seconds.
Conclusion

- Interval Hungarian Algorithm.
- Measures the effect of uncertainty on the optimality and stability of output.
- Data from simulation and real robot experiments are provided.
Thank you!

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