Prediction Market, Mechanism Design, and Cooperative Game Theory

V. Conitzer

presented by Janyl Jumadinova
October 16, 2009
Predictive Markets

- Created for the purpose of making predictions by obtaining information from multiple agents
- **Good** prediction market guarantees all agents are rewarded for useful and accurate information
- Each agent has a subjective probability about a particular event
- Agent buys if the price < subjective probability, and sell otherwise
Most assume agents are myopic

However strategic agents do not always behave myopically

Strategic agents can act in hopes of maximizing their long term profits
Proposed Solution

- Mechanism design - designing efficient markets irrespective of strategic behavior
- Mechanism design approach: design direct-revelation incentive compatible mechanism: agents directly report all their private information
- Goal is to reduce a gap between theories of prediction markets and mechanism design
- Design direct-revelation market based on proper scoring rules
- Propose several specific mechanisms
Proper Scoring Rules

- Incentivizes a single agent to truthfully report its subjective probability $p_E$ that an event $E$ will take place.
- Pay the agent the amount depending on both reported probability and whether event actually occurs.
- Let: $x_E = 1$ if the event occurs, $x_E = 0$ otherwise, $\hat{p}_E$ - reported probability, $s(\hat{p}_E, x_E)$ - agent’s payment.
- $s$ is a proper scoring rule if the agent maximizes its subjective expected payoff by giving its true estimate of the probability.
- $\forall p \in [0, 1], \{p\} = \arg \max_{\hat{p}} ps(\hat{p}, 1) + (1 - p)s(\hat{p}, 0)$
- Examples: $s(\hat{p}_E, x_E) = 1 - (x_E - \hat{p}_E)^2$ (quadratic), $s(\hat{p}_E, x_E) = x_E \log \hat{p}_E + (1 - x_E) \log(1 - \hat{p}_E)$ (logarithmic).
Mechanism Design

- $n$ agents with private information (signal) $\theta_i \in \Theta_i$, where $\Theta_i$ is the set of signals $i$ might receive
- Agent $i$ reports some $\hat{\theta}_i \in \Theta_i$ (not necessarily its true signal)
- The mechanism is defined as: $f : \Theta_1 \times \Theta_2 \times \ldots \times \Theta_n \rightarrow O$, where $O$ is the set of all possible outcomes in the domain
- Each agent $i$ has a utility function $u_i : \Theta \times O$, where $\Theta = \Theta_1 \times \ldots \times \Theta_n$
- $u_i(\theta, o)$ - agent $i$’s utility when agent’s true signals are $\theta$ and the outcome chosen is $o$
Mechanism Design

- *Incentive compatible* - mechanism that incentivize agents to report their signals truthfully

- *Ex-post incentive compatible* if
  \[ u_i((\theta_i, \theta_{-i}), f(\theta_i, \theta_{-i})) \geq u_i((\theta_i, \theta_{-i}), f(\hat{\theta}_i, \theta_{-i})), \]
  \[ \forall i, \theta_i, \hat{\theta}_i \in \Theta_i, \theta_{-i} \in \Theta_{-i} \]

- Each agent is better off reporting truthfully regardless of the signals the other agents receive

- *Ex-interim incentive comparable* - must have prior distribution over the joint signal space \( \Theta \)
  \[ E_{\theta_{-i}}[u_i((\theta_i, \theta_{-i}), f(\theta_i, \theta_{-i}))] \geq E_{\theta_{-i}}[u_i((\theta_i, \theta_{-i}), f(\hat{\theta}_i, \theta_{-i}))], \]
  \[ \forall i, \theta_i, \hat{\theta}_i \in \Theta_i, \theta_{-i} \in \Theta_{-i} \]

- Each agent is in expectation over the others' signals better off reporting truthfully regardless of the signals the other agents receive
Cooperative game theory

CHARACTERISTIC FUNCTION GAME:

- Set of agents 1, ..., n, characteristic function $v: 2^{\{1,\ldots,n\}} \rightarrow \mathbb{R}$
- For any subset (coalition) $C$ of the agents, $v(C)$ is the value that $C$ can generate by working together
- **Grand coalition** - all agents
- How the total value generated by the grand coalition $v(\{1,\ldots,n\})$ should be distributed among the agents?
Characteristic function game

- Impose ordering on the agents using permutation $\pi$
- $\pi(j)$ is the agent ranked $j$th in the ordering
- Give agent $i$ ranked in the $\pi^{-1}(i)$ the position, its marginal contribution
  
  Marginal contribution is
  
  $$\nu(\{\pi(1), \pi(2), \ldots, \pi(\pi^{-1}(i))\}) - \nu(\{\pi(1), \pi(2), \ldots, \pi(\pi^{-1}(i)) - 1\}),$$
  
  $\pi(\pi^{-1}(i)) = i$

- If there is no natural order on agents - Shapley Value
  
  Agent $i$ gets
  
  $$\frac{1}{n!} \sum_{\pi} \nu(\{\pi(1), \pi(2), \ldots, \pi(\pi^{-1}(i))\}) - \nu(\{\pi(1), \pi(2), \ldots, \pi(\pi^{-1}(i)) - 1\})$$
State-based model for prediction markets

- $S$ - set of states the world can be in, $P$ is the common prior over these states
- Want to assess the probability of some event $E \subseteq S$ given agents’ information
- Each agent $i$ has private information $S_i \subseteq S$
- Agent $i$ has to report its full information $S_i$ directly
- No new information enters the market
- $\bigcap_i S_i$ is the combined information of all agents
- $P(E | \bigcap_i S_i)$ - conditional probability for the event $E$
Two-state example with $S = \{a, b\}$, where $E$ is true for $a$ and false for $b$ (so that we are predicting the probability that $a$ happens), and the prior is uniform. Suppose that there is only a single agent.

If the true state is $a$, then with probability .5, the agent receives the signal $\theta = \{a\}$ (she can rule out state $b$), and with probability .5, she receives the signal $\theta = \{a, b\}$ (she cannot rule out any state).

If the true state is $b$, then with probability 1 she receives the signal $\theta = \{a, b\}$.

The signal is always consistent with the true state, and none of the states in a signal can be ruled out.

However, we have $P(E|\theta = \{a, b\}) = P(E \land [\theta = \{a, b\}])/P(\theta = \{a, b\}) = (1/4)/(1/4 + 1/2) = 1/3$.

In contrast, we have $P(E|\{a, b\}) = P(E) = 1/2$ (because $P(E|\{a, b\})$ is the probability that $E$ happens given that the true state is $a$ or $b$, which is always true).
State-based model for prediction markets: Definitions

**Definition 1** The model is consistent if, for every subset $C$ of the agents, for every combination $\theta_C$ of $|C|$ signals that the agents in $C$ can receive (where $\theta_i = S_i \subseteq S$), we have $P(E|\theta_C) = P(E|\cap_{i \in C} S_i)$.

We now show that the inconsistency in Example 1 is due to the randomness of the signals.

**Definition 2** We say that signals are deterministic if for every agent $i$, there is a partition $S_i^1, S_i^2, \ldots, S_i^{k_i}$ of $S$, so that if the true state is $s \in S_i^j$, then agent $i$ is guaranteed to get signal $S_i^j$.

**Proposition 1** If signals are deterministic, then the model is consistent.
To make sure that signals are deterministic, make them part of the state space.

**Example 2** Consider Example 1. We extend the state space to have three states: \(a_1 = (a \land [\theta = \{a\}]), a_2 = (a \land [\theta = \{a, b\}]), b_1 = (b \land [\theta = \{a, b\}]). These states happen with probabilities 1/4, 1/4, 1/2, respectively. We have \(E = \{a_1, a_2\}. In this modified state space, the signal \(\theta'\) that the agent receives is either \(\{a_1\}\) or \(\{a_2, b_1\}\), so signals are deterministic. Indeed, \(P(E|\theta' = \{a_2, b_1\}) = P(E|\{a_2, b_1\}) = (1/4)/(1/4 + 1/2) = 1/3.\)
Specific Mechanisms for rewarding the agents for their information

**Rewarding agents individually**

- Given common prior $P$, each agent's reported individual information $\hat{S}_i$ leads to a probability estimate $P(E|\hat{S}_i)$ for which the agent can be rewarded using the proper scoring rule

**Definition 3** Under the individual-rewarding information mechanism, agent $i$ receives $s(P(E|\hat{S}_i), x_E)$.

**Proposition 2** If the model is consistent, then the individual-rewarding information mechanism is ex-interim incentive compatible.
Rewarding agents individually

Disadvantages:

- Reporting truthfully is not uniquely optimal action and not ex-post incentive compatible
- May result in payments to agents that do not contribute any information
- Not rewarding more valuable information more, maybe paying twice for the same information
Rewarding agents based on marginal information

Pay only for the marginal information agent reports

**Definition 4** If the agents are ordered 1, ..., n, then under the marginal information mechanism, agent i receives

\[ s(P(E|\cap_{j=1,\ldots,i} \hat{S}_j), x_E) - s(P(E|\cap_{j=1,\ldots,i-1} \hat{S}_j), x_E) \]

Can be viewed as a special case of market scoring rule

- If agent shifts market probability estimate \( \hat{p}_E \) to \( \hat{p}_E' \), it receives
  \[ s(\hat{p}_E', x_E) - s(\hat{p}_E, x_E) \]
- If each agent can change the market probability only once in order 1, .., n, if \( \hat{p}_E^i \) is the marginal probability after i’s move, then agent i receives
  \[ s(\hat{p}_E^i, x_E) - s(\hat{p}_E^{i-1}, x_E) \]
Rewarding agents based on marginal information

Advantages:
- The final probability reflects all the information available to the agents (direct revelation)
- Agents don’t need to reason about how probability should be updated
- Pays exactly for the total information it receives

Proposition 3 If the model is consistent, then the marginal information mechanism is ex-interim incentive compatible.
May not be a natural order on the agents to use in the marginal information mechanism

**Definition 5**  *Under the Shapley value information mechanism, agent $i$ receives the average, taken over all orders of the agents, of what she would have received under the marginal information mechanism.*

Because the Shapley value information mechanism is simply the average of all marginal information mechanisms, it also pays exactly $s(P(E|\bigcap_{j=1,...,n}\hat{S}_j), x_E) - s(P(E|S), x_E)$ in the end.

**Proposition 4**  *If the model is consistent, the Shapley value information mechanism is ex-interim incentive compatible.*
The value generated by a coalition $C \subseteq \{1, ..., n\}$ in the absence of other agents is $s(P(E|\bigcap_{j \in C} \hat{S}_j), x_E) - s(P(E|C), x_E)$

Can use any solution concept from cooperative game theory
Conversion to characteristic function games

- **Pre-characteristic function** $\pi : 2^{\{1,\ldots,n\}} \rightarrow [0, 1]$, where
  \[ \pi(C) = P(E | \bigcup_{j \in C} \hat{S}_j) \]

- Theorem 1 shows that this function can take any form as long as probabilities are not equal to 0 or 1

**Theorem 1** For any function $\psi : 2^{\{1,\ldots,n\}} \rightarrow (0, 1)$, there exists an instance such that for all $C$, 
\[ \pi(C) = P(E | \bigcap_{j \in C} \hat{S}_j) = \psi(C). \]
Rewarding based on the group’s information

Ex-post incentive compatible mechanism

**Definition 6** Under the group-rewarding information mechanism, every agent $i$ receives the same amount, $s(P(E | \bigcap_{j=1,...,n} \hat{S}_j), x_E)$.

**Proposition 5** If the model is consistent, then the group-rewarding information mechanism is ex-post incentive compatible.

Agent who reports no information gets paid the same as the agent who reports a lot of information
Modification of the group-rewarding information mechanism

**Definition 7** Under the pivotal information mechanism, agent $i$ receives $s(P(E|\bigcap_{j=1}^{n} \hat{S}_j), x_E) - s(P(E|\bigcap_{j\neq i} \hat{S}_j), x_E)$.

**Proposition 6** If the model is consistent, then the pivotal information mechanism is ex-post incentive compatible.
Definition 8 A mechanism satisfies the individual agent property if, for every \( i \), we have the following: given that the other agents report no information (\( \hat{S}_j = S \) for all \( j \neq i \)), agent \( i \) receives \( s(P(E|S_i), x_E) - s(P(E|S), x_E) \).

Definition 9 A mechanism satisfies the strong decomposition property if, for every \( i \), for every \( S', S'', S''' \subseteq S \), we have the following: if agent \( i \) reports \( \hat{S}_i = S' \cap S'' \), and the other agents report \( \bigcap_{j \neq i} \hat{S}_j = S''' \), then the reward that \( i \) gets is equal to the reward that she would get if she reported \( S'' \) and the other agents reported \( S''' \), plus the reward that she would get if she reported \( S' \) and the other agents reported \( S'' \cap S''' \). It satisfies the weak decomposition property if the above holds when \( S''' = S \).

Definition 10 A mechanism satisfies the relative dummy property if, for every \( i \), given that \( i \) reports no information that is not also reported by the other agents (that is, \( \bigcap_{j \neq i} \hat{S}_j \subseteq \hat{S}_i \)), agent \( i \) receives 0.
Lemma 1 The weak decomposition property implies the relative dummy property.

Theorem 2 The pivotal information mechanism satisfies the individual agent, strong decomposition, and relative dummy properties. No other mechanism satisfies the individual agent and weak decomposition properties.
Creating a practical design with information and probability agents

- Directly requires a model of all the information that can be reported of all the possible states of the world and a prior distribution over these states
- Assume have to types of agents: *information* and *probability* agents
- Information agents have relevant information about the event trying to predict
- Probability agents are able to convert any information that is given to them into a probability
Creating a practical design with information and probability agents

- To run mechanism, given a collection of information (restricted set of states), compute conditional probability of the event happening.
- Use *probability* agents to get estimates for conditional probabilities using a standard prediction market.
Creating a practical design with information and probability agents

```
Initialize \( \hat{P}_E \) to some value
\( \hat{P}_0(E) \leftarrow \hat{P}_E \)
for \( j = 1 \) to \( n_2 \) {
    Probability agent \( j \) moves \( \hat{P}_E \) to a new value
    \( \hat{P}_j(E) \leftarrow \hat{P}_E \)
}
\( \hat{P}(E) \leftarrow \hat{P}_{n_2}(E) \)
for \( i = 1 \) to \( n_1 \) {
    Information agent \( i \) reports information \( I_i \) (in natural language)
    \( \hat{P}_0(E|I_1, \ldots, I_i) \leftarrow \hat{P}_E \)
    for \( j = 1 \) to \( n_2 \) {
        Probability agent \( j \) moves \( \hat{P}_E \) to a new value
        \( \hat{P}_j(E|I_1, \ldots, I_i) \leftarrow \hat{P}_E \)
    }
    \( \hat{P}(E|I_1, \ldots, I_i) \leftarrow \hat{P}_{n_2}(E|I_1, \ldots, I_i) \)
}

The event is realized to obtain \( x_E \)
for \( i = 1 \) to \( n_1 \)
    Reward information agent \( i \) with \( s_1(\hat{P}(E|I_1, \ldots, I_i), x_E) - s_1(\hat{P}(E|I_1, \ldots, I_{i-1}), x_E) \)
for \( j = 1 \) to \( n_2 \)
    Reward probability agent \( j \) with \( \sum_{i=0}^{n_1} [s_2(\hat{P}_j(E|I_1, \ldots, I_i), x_E) - s_2(\hat{P}_{j-1}(E|I_1, \ldots, I_i), x_E)] \)
```
Conclusion and Future work

- Proposed mechanism design based model for prediction markets
- Maybe unnatural to implement
- New information mechanisms
- Study properties of mechanisms
- Implementation of the mechanism